THE GAUSS MAP AND A NONCOMPACT RIEMANN-ROCH FORMULA FOR CONSTRUCTIBLE SHEAVES ON SEMIABELIAN VARIETIES

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Abstract: For an irreducible subvariety Z in an algebraic group G we define an integer $\mathrm{gdeg}(Z) \geq 0$ as the degree, in a certain sense, of the Gauss map of Z. It can be regarded as a substitution for the intersection index of the conormal bundle to Z with the zero section of T^*G, even though G may not be compact. For G a semiabelian variety (in particular, an algebraic torus $(\mathbb{C}^*)^n$), we prove a Riemann-Roch type formula for constructible sheaves on G which involves our substitutions for the intersection indicies. As a corollary, we get that a perverse sheaf on such a G has nonnegative Euler characteristic, generalizing a theorem of Loeser-Sabbah.

1. Introduction and statement of results

Let X be a smooth algebraic variety over $\mathbb{C}$ and $\mathfrak{F}$ be a constructible sheaf of $\mathbb{C}$-vector spaces on X. As in other situations, we have the Riemann-Roch problem: express $\chi(X, \mathfrak{F}) = \sum_i (-1)^i \dim H^i(X, \mathfrak{F})$ in terms of some intrinsic geometric invariants of X and $\mathfrak{F}$. One such invariant is the characteristic cycle $CC(\mathfrak{F})$ which is a formal $\mathbb{Z}$-linear combination $\sum_\nu n_\nu [\Lambda_\nu]$ of irreducible conic Lagrangian subvarieties Λ_ν in the cotangent bundle T^*X, see [14] for background. When X is compact, the Riemann-Roch problem has a nice solution, namely [13]:

$$\chi(X, \mathfrak{F}) = (CC(\mathfrak{F}), [X])_{T^*X}, \tag{1.1}$$

where the right hand side is the intersection index, in T^*X, of $CC(\mathfrak{F})$ and the zero section $X \subset T^*X$. It can be calculated, for instance, by deforming X to the graph of a C^∞ 1-form so that the intersection becomes transverse, and then counting intersection points (with multiplicities and signs).

Both the definition of $CC(\mathfrak{F})$ and the formula (1.1) (for compact X) extend to the case when $\mathfrak{F}$ is a bounded constructible complex (i.e., a complex of sheaves with constructible cohomology).

When X is not compact, $\chi(X, \mathfrak{F})$ still makes sense, but (1.1) is not applicable. We face, therefore, an interesting *noncompact Riemann-Roch problem* of finding $\chi(X, \mathfrak{F})$ in terms of invariants intrinsic to X (in particular, not involving the choice of compactification).

The purpose of this paper is to exhibit such a "noncompact Riemann-Roch formula" in a particular class of situations. Namely, suppose that $X = G$ is an algebraic group with Lie algebra $\mathfrak{g}$. For $\gamma \in \mathfrak{g}^*$, let ω_γ be the corresponding left-invariant 1-form on G, and $\Omega_\gamma \subset T^*G$ be its graph. The Ω_γ then form a natural family of deformations of X and we can use them to make sense of the intersection

index in (1.1) even when G is not compact. More precisely, if $\Lambda \subset T^*G$ is an irreducible conic Lagrangian subvariety and $\gamma \in \mathfrak{g}^*$ is *generic*, then $\Lambda \cap \Omega_\gamma$ consists of finitely many transversal intersection points; their number will be denoted $\mathrm{gdeg}(\Lambda)$ and called the *Gaussian degree* of Λ. To explain the name, recall that Λ has the form $T^*_Z X$ for an irreducible subvariety $Z \subset X$ (notation: $T^*_Z X$ will always mean the closure of the conormal bundle to the smooth locus Z_{sm} of Z). Denoting $k = \dim(Z)$, we have the left Gauss map

$$\Gamma_Z : Z \to G(k, \mathfrak{g}) \ , \ z \mapsto z^{-1}(T_z Z) \subset T_e G = \mathfrak{g}$$

which is a rational map, regular on Z_{sm}. The number $\mathrm{gdeg}(\Lambda)$ is the degree of Γ_Z in an appropriate sense, see section 2. For example, if Z is a hypersurface, then the source and the target of Γ_Z have the same dimension, and $\mathrm{gdeg}(\Lambda)$ is the degree of Γ_Z in the usual sense.

Note that $\mathrm{gdeg}(\Lambda) \geq 0$ by construction. We now formulate the main result of this note. Recall [2] that a semiabelian variety is an algebraic group G which is an extension

$$1 \to T \to G \to A \to 1, \tag{1.2}$$

where A is an abelian variety and $T \cong (\mathbb{C}^*)^n$ is an algebraic torus.

(1.3) Theorem. *Let G be a semiabelian variety, $\mathfrak{F}$ a bounded constructible complex on G and $CC(\mathfrak{F}) = \sum_\nu n_\nu [\Lambda_\nu]$. Then*

$$\chi(G, \mathfrak{F}) = \sum_\nu n_\nu \mathrm{gdeg}(\Lambda_\nu).$$

(1.4) Corollary. *If, in the situation of (1.3), $\mathfrak{F}$ is a perverse sheaf, then $\chi(G, \mathfrak{F}) \geq 0$.*

Indeed, for $\mathfrak{F}$ a perverse sheaf, all $n_\nu \geq 0$. Here we use the conventions of [1] for the definition of (middle) perversity.

(1.5) Corollary. *If G is semiabelian and $Z \subset G$ is a smooth closed subvariety, then the number $(-1)^{\dim(Z)} \chi(Z, \mathbb{C})$ is nonnegative and coincides with $\mathrm{gdeg}(Z)$.*

Indeed, $\underline{\mathbb{C}}_Z[\dim(Z)]$ is perverse and its characteristic cycle is $[T^*_Z G]$ taken with multiplicity 1.

Corollary 1.4 for $G = (\mathbb{C}^*)^n$ was proven by Loeser-Sabbah [12] and given a different proof (applicable to etale sheaves) by Gabber-Loeser [4]. In the case when G is an abelian variety, Corollary 1.4 seems to be new, even though Theorem 1.3 in this case does not need a special proof, being a consequence of (1.1). Moreover, since the general proof of Theorem 1.3 given below is extremely simple and transparent, we believe that our approach exhibits the true reason behind the Loeser-Sabbah observation. Further, another result of [4] identifies irreducible perverse sheaves $\mathfrak{F}$ on $(\mathbb{C}^*)^n$ such that $\chi((\mathbb{C}^*)^n, \mathfrak{F}) = 1$, with complexes of solutions of hypergeometric systems (essentially, of the A-hypergeometric systems [5][11], see [6] for a comparison of the two points of view). On the other hand, in [12] the second author classified irreducible hypersurfaces in $(\mathbb{C}^*)^n$ for which the Gauss map has degree 1 and identified them with (reduced) A-discriminantal hypersurfaces. The

latter describe the characteristic varieties of the A-hypergeometric system. Thus our approach explains the analogy between these two results.

Unfortunately, Theorem 1.3 cannot be straightforwardly generalized to more general algebraic groups. For example, if G is nonabelian reductive, then it contains affine spaces $\mathbb{A}^m$ for m ranging from 0 to the number of positive roots of G. The Euler characteristic of $\mathbb{A}^m$ being 1, Corollary 1.5 (and thus Theorem 1.3) cannot hold. The same applies when G, while commutative, contains G_a, the additive group. Nevertheless, we believe that there should exist generalizations of Theorem 1.3 which involve some particular classes of constructible sheaves and complexes.

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2. The Gauss map and the Gaussian degree

Let G be a complex algebraic group, $\mathfrak{g}$ its Lie algebra and $G(k, \mathfrak{g})$ the Grassmannian of k-dimensional linear subspaces in $\mathfrak{g}$. If $Z \subset G$ is an irreducible k-dimensional subvariety, we have the rational map

$$\Gamma_Z : Z \to G(k, \mathfrak{g})$$

called the (left) Gauss map and defined as follows. For $x \in G$ let

$$l_x : G \to G \ , \ l_x(y) = xy$$

be the left translation by x. Then, for a smooth point $z \in Z$ the value $\Gamma_Z(z)$ is the image of

$$d_x l_{x^{-1}} : T_z Z \to T_e G = \mathfrak{g}.$$

We want to associate to Γ_Z a nonnegative integer called its degree. To this end, consider first a more general situation.

Let M be an irreducible k-dimensional variety, V an n-dimensional vector space and $f : M \to G(k, V)$ a rational map. Replacing, if necessary, M with its Zariski open subset, we can assume that f is regular. Consider the flag variety $F(k, n-1, V)$ and its projections

$$G(k, V) \xleftarrow{p} F(k, n-1, V) \xrightarrow{q} G(n-1, V) = P(V^*)$$

Let $\widetilde{M} = G(k, V) \times_M F(k, n-1, V)$ be the fiber product with respect to f and p. Since p is a smooth map with $(n-k-1)$-dimensional fibers, $\widetilde{M}$ is an irreducible variety of dimension $n-1$. The map q induces a regular map $q' : \widetilde{M} \to \mathbb{P}(V^*)$ whose source and target have the same dimension. We define $\deg(f)$ to be the degree of the map q'. The following is then clear.

(2.1) Proposition. *If $W \subset V$ is a generic hyperplane, then $\deg(f)$ is equal to the number of $x \in M$ such that $f(x) \in G(k, W) \subset G(k, V)$. For any such x the map f is locally (in the analytic or etale cohomology) an embedding near x and the intersection of $f(M)$ and $G(k, W)$ is transversal at x.*

When $k = n-1$, we have $\widetilde{M} = M$ and $\deg(f)$ is the degree of f in the usual sense.

We now specialize to the case $M = Z$, $V = \mathfrak{g}$, $n = \dim(G)$ and $f = \Gamma_Z$. The number $\deg(\Gamma_Z)$ will be denoted by $\mathrm{gdeg}(Z)$ and called the Gaussian degree of Z.

Let $\Lambda = T^*_Z G$ be the conic Lagrangian variety associated to Z. We will write $\mathrm{gdeg}(\Lambda)$ for $\mathrm{gdeg}(Z)$. As in Section 1, for $\gamma \in \mathfrak{g}^*$ let $\Omega_\gamma \subset T^*X$ be the graph of the left-invariant1-form ω_γ on G associated to γ. Proposition 2.1 implies easily:

(2.2) Proposition. *Let $\gamma \in \mathfrak{g}^*$ be a generic linear functional. Then $\Lambda \cap \Omega_\gamma$ consists of finitely many points which are smooth on Λ and in which the intersection is transverse. The number of these points is equal to* $\mathrm{gdeg}(\Lambda)$.

3. Characteristic cycle of an open embedding

A non-intrinsic way to find $\chi(U, \mathfrak{F})$ where U is a noncompact manifold, is to apply (1.1) to $Rj_*\mathfrak{F}$ where $j : U \hookrightarrow X$ is a smooth compactification. We will indeed use this approach in the proof of Theorem 1.3, so we recall the (now well known) proceedure of finding $CC(Rj_*\mathfrak{F})$ from $CC(\mathfrak{F})$, see [7] [17].

Let X be a not necessarily compact smooth variety, $f \in \mathbb{C}[X]$ a regular function, $U \subset X$ the open set $\{f \neq 0\}$, and $j : U \hookrightarrow X$ the embedding. Let $\Lambda \subset T^*U$ be an irreducible conic Lagrangian variety. For $s \in \mathbb{C}^*$ let

$$\Lambda_s^\# = \Lambda + s\, d\log f = \{(\xi + s(d\log f)(x), x) | (x, \xi) \in \Lambda\} \tag{3.1}$$

This is a closed (no longer conic) Lagrangian subvariety in T^*X. The total space of the family of $\Lambda_s^\#$ is a subvariety $\Lambda^\# \subset T^*X \times \mathbb{C}^*$. The limit $\lim\limits_{s\to 0} \Lambda_s^\#$ (also called the specialization of $\Lambda^\#$ in [7]) is an effective Lagrangian cycle in T^*X defined as follows. We first take the closure $\overline{\Lambda^\#}$ in $T^*X \times \mathbb{C}$ and then form the scheme-theoretic intersection $\overline{\Lambda^\#} \cap (T^*X \times \{0\})$. The cycle $\lim\limits_{s\to 0} \Lambda_s^\#$ is obtained by taking the irreducible components of this intersection with the multiplicities given by the scheme structure.

We extend this construction by $\mathbb{Z}$-linearity to conic Lagrangian cycles in T^*U. Thus, if Σ is such a cycle, we have the family of non-conic cycles $\Sigma_s^\#$, $s \in \mathbb{C}^*$ and the conic cycle $\lim\limits_{s\to 0} \Sigma_s^\#$ in T^*X. Now, the fact we need is as follows.

(3.2) Theorem. *If $\mathfrak{F}$ is a bounded constructible complex on U, then*

$$CC(Rj_*\mathfrak{F}) = \lim_{s\to 0} CC(\mathfrak{F})_s^\#.$$

This statement can be obtained from Theorem 3.2 of [7] by applying the Riemann-Hilbert correspondence, or from Theorem 3.1 of [17] which is applicable to the more general case of $\mathbb{R}$-constructible sheaves. (To be precise, the concepts of the characteristic cycle used in [7-9] and [13][14][17] refer to different contexts: holonomic D-modules vs. constructible complexes. The compatibility of these two definitions of the characteristic cycle inder the Riemann-Hilbert correspondence follows from the results of [7].)

Consider now a nominally more general situation (cf. [9], Appendix A): let X be as before but suppose that we have n regular functions $f_1, \ldots, f_n \in \mathbb{C}[X]$. Let U be the intersection of the n open sets $\{f_i \neq 0\}$ and $j : U \hookrightarrow X$ be the embedding. Of course, this situation can be analyzed by applying Theorem 3.2 to $f = f_1 \ldots f_n$, but it will be convenient for us to have a more flexible formulation.

For a point $s = (s_1, \ldots, s_n) \in (\mathbb{C}^*)^n$ and a conic Lagrangian variety $\Lambda \subset T^*U$ we form, similarly to (3.1), a non-conic Lagrangian variety

$$\Lambda_s^{\#} = \Lambda + \sum_{i=1}^{n} s_i d \log f_i \subset T^*X. \tag{3.3}$$

The total space of this family lies in $T^*X \times (\mathbb{C}^*)^n$. Taking the closure in $T^*X \times \mathbb{C}^n$ and then intersecting with $T^*X \times \{(0, \ldots, 0)\}$ defines, similarly to the above, a conic Lagrangian cycle $\lim\limits_{s \to (0,\ldots,0)} \Lambda_s^{\#}$. Of course, this "limit" could be taken along any curve in $\mathbb{C}^n$ passing through 0 and generically lying in $(\mathbb{C}^*)^n$. As before, we extend this construction by linearity to conic Lagrangian cycles in T^*U. The next statement follows by iterated application of Theorem 3.2.

(3.4) Theorem. *If $\mathfrak{F}$ is a bounded constructible complex on U, then*

$$CC(Rj_*\mathfrak{F}) = \lim_{s \to (0,\ldots,0)} CC(\mathfrak{F})_s^{\#}\ ,\ s = (s_1, \ldots, s_n) \in (\mathbb{C}^*)^n.$$

Taking the limit along different curves approaching $(0, \ldots, 0)$ corresponds, roughly, to different choices of our equation for the reducible hypersurface $X \setminus U$. For example, restricting to the curve with parametric equation $s_i = t^{m_i}$, $m_i > 0$ corresponds to taking $\prod_i f_i^{m_i}$ as an equation.

We now need a slight globalization of Theorem 3.4. First of all, let (L, ∇) be a line bundle on X with an algebraic flat connection. If f is a regular section of L, then $f^{-1}\nabla f$ is a scalar 1-form regular over the open set $\{f \neq 0\}$. We denote this form $\nabla \log f$.

Suppose now that we have n line bundles with flat connections (L_i, ∇_i) on X, $i = 1, \ldots, n$. Suppose $f_i \in \Gamma(X, L_i)$, $i = 1, \ldots, n$ and $U \subset X$ is the intersection of the open sets $\{f_i \neq 0\}$. As before, let $j : U \hookrightarrow X$ be the embedding.

(3.5) Theorem. *For a bounded constructible complex $\mathfrak{F}$ on U we have*

$$CC(Rj_*\mathfrak{F}) = \lim_{s \to (0,\ldots,0)} \left(CC(\mathfrak{F}) + \sum_{i=1}^{n} s_i \nabla_i \log f_i \right).$$

Proof: As before, it is enough to consider the case $n = 1$, as the general case can be obtained by iteration, as in [9], Appendix A. The statement for $n = 1$ is a consequence of Theorem 6.3 of [8] which deals with the more general case of the zero locus of a section f of an arbitrary line bundle L, not necessarily with connection. The recipe in this case is to consider the "twisted cotangent bundles" $(T^*X)^{(s)}, s \in \mathbb{C}$, defined as the symplectic quotients of T^*L by the hamiltonian action of $\mathbb{C}^*$ induced by dilations of L. Now, if L is equipped with a flat connection ∇, then all the $(T^*X)^{(s)}$ become identified with T^*X and the formulation of [8], Theorem 6.3 reduces to our statement.

4. Proof of Theorem 1.3 for $G = (\mathbb{C}^*)^n$

We first consider the case when $G = (\mathbb{C}^*)^n$ is an algebraic torus. Let $z_1, \ldots, z_n$, $z_i \neq 0$, be the standard coordinates in $(\mathbb{C}^*)^n$. We compactify G by the projective space $\mathbb{P}^n$ with homogeneous coordinates $(t_0 : \cdots : t_n)$ by

$$j : (\mathbb{C}^*)^n \hookrightarrow \mathbb{P}^n\ ,\ (z_1, \ldots, z_n) \mapsto (1 : z_1 : \cdots : z_n)$$

For $\nu = 0, \dots, n$ let $A^n_\nu \subset \mathbb{P}^n$ be the affine chart given by $t_\nu \neq 0$. This is an affine space with coordinates $z_i^{(\nu)}$, $i \in \{0, \dots, n\} \setminus \{\nu\}$ given by $z_i^{(\nu)} = \frac{t_i}{t_\nu}$. Denote by

$$(\mathbb{C}^*)^n \overset{j_\nu}{\hookrightarrow} A^n_\nu \overset{k_\nu}{\hookrightarrow} \mathbb{P}^n$$

the embeddings. For $\nu = 0$ we have $z_i^{(0)} = z_i$.

We now apply Theorem 3.4 to $U = (\mathbb{C}^*)^n$, $X = A^n_0$, $f_i = z_i$, and our constructible complex $\mathfrak{F}$. The recipe of the theorem requires us to introduce the family of 1-forms

$$\omega_s = \sum_{i=1}^n s_i \, d\log z_i \ , \ s = (s_1, \dots, s_n) \in (\mathbb{C}^*)^n.$$

These forms are precisely the invariant 1-forms on $G = (\mathbb{C}^*)^n$. We can view s as an element of $\mathfrak{g}^*$, where $\mathfrak{g} = \mathbb{C}^n$ is the Lie algebra of G. Theorem 3.4 then gives us:

$$CC(Rj_{0*}\mathfrak{F}) = CC(Rj_*\mathfrak{F})|_{A^n_0} = \lim_{s \to 0} (CC(\mathfrak{F}) + \omega_s), \tag{4.1}$$

the limit being taken in $T^*A^n_0$.

Next, we apply Theorem 3.4 to $U = (\mathbb{C}^*)^n$ and $X = A^n_\nu$ with arbitrary $\nu \in \{0, \dots, n\}$. Then we should take $f_i = z_i^{(\nu)}$, $i \in \{0, \dots, n\} \setminus \{\nu\}$ and consider the 1-forms

$$\omega^{(\nu)}_{s'} = \sum_{i \neq \nu} s'_i \, d\log z_i^{(\nu)} \ , \ s' \in (\mathbb{C}^*)^{\{0,\dots,n\}\setminus\{\nu\}}.$$

Now, each $z_i^{(\nu)}$ is a Laurent monomial in the $z_1, \dots, z_n$, so $d\log z_i^{(\nu)}$ is an (integer) linear combination of $d\log z_1, \dots, d\log z_n$. Therefore, $\omega^{(\nu)}_{s'} = \omega_s$ where $s = \phi_\nu(s')$ is an image of s under a linear transformation $\phi_\nu : \mathbb{C}^{\{0,\dots,n\}\setminus\{\nu\}} \to \mathbb{C}^n$. It is clear that ϕ_ν is invertible; in particular, $s \to 0$ iff $s' \to 0$. This means that the answers given by Theorem 3.4 for the $CC(Rj_{\nu*}\mathfrak{F})$, glue together into one global answer:

$$CC(Rj_*\mathfrak{F}) = \lim_{\substack{s \to 0 \\ s \in E}} (CC(\mathfrak{F}) + \omega_s). \tag{4.2}$$

Here the limit is taken in $T^*\mathbb{P}^n$ and s runs over the set

$$E = \{(s_1, \dots, s_n) \in (\mathbb{C}^*)^n : \ s_i \neq s_j \ , \ i \neq j\}.$$

(The condition $s \in E$ is equivalent to $\phi_\nu^{-1}(s) \in (\mathbb{C}^*)^{\{0,\dots,n\}\setminus\{\nu\}}$ for any ν.)

Now, Theorem 1.3 for $G = (\mathbb{C}^*)^n$ would follow from (4.2) and the next lemma.

(4.3) Lemma. *Let $\Lambda \subset T^*(\mathbb{C}^*)^n$ be an irreducible conic Lagrangian variety. Then*

$$\lim_{\substack{s \to 0 \\ s \in E}} (\Lambda + \omega_s), [\mathbb{P}^n])_{T^*\mathbb{P}^n} = \mathrm{gdeg}(\Lambda).$$

Proof: Let $\Omega_s \subset T^*(\mathbb{C}^*)^n$ be the graph of ω_s. If $s \in E$, then Ω_s is closed in $T^*\mathbb{P}^n$ as well. By translation, intersecting $\Lambda + \omega_s$ with $[\mathbb{P}^n]$ is equivalent to intersecting Λ with Ω_{-s}. By Proposition 2.2, there exists a Zariski open, nonempty set $F \subset \mathbb{C}^n$ such that for $s^{(0)} \in F$ the intersection $\Lambda \cap \Omega_{-s^{(0)}}$ consists of $\mathrm{gdeg}(\Lambda)$ smooth transverse points. Since E is also Zariski open, $F \cap E$ meets any polydisk

$$P_\varepsilon = \{(z_1, \dots, z_n) \in (\mathbb{C}^*)^n : \ 0 < |z_i| < \varepsilon\}$$

in an open dense set. By the "continuity of intersection" it follows that for $|\varepsilon| \ll 1$ and $s^{(0)} \in P_\varepsilon \cap F \cap E$

$$\left(\lim_{\substack{s \to 0 \\ s \in E}} \Lambda + \omega_s), [\mathbb{P}^n] \right)_{T^*\mathbb{P}^n} = |\Omega_{-s^{(0)}} \cap \Lambda| = \mathrm{gdeg}(\Lambda)$$

and this completes the proof.

Proof of Theorem 1.3 in general

Let A be an abelian variety. We recall some elementary properties of line bundles on A, see [10] [16]. If L is such a bundle, by L_a, $a \in A$ we denote its fiber at a. By L^0 we denote the total space of L with the zero section deleted, so L^0 is a principal $\mathbb{C}^*$-bundle on A. As for any base variety, the correspondence $L \mapsto L^0$ is an equivalence between the category of line bundles and isomorphisms and the category of principal $\mathbb{C}^*$-bundles.

Assume that L has degree 0. Then the theorem of the square [16] provides identifications

$$L_0 \cong \mathbb{C}\ ,\ L_a \otimes L_b \widetilde{\longrightarrow} L_{a+b}$$

which make L^0 into a group, namely a semiabelian variety fitting into an extension

$$0 \to \mathbb{C}^* \to L^0 \xrightarrow{p} A \to 0. \tag{5.1}$$

Next, any bundle L of degree 0 has a flat connection. All such connections form an affine space $\mathrm{Conn}(L)$ over the vector space $H^0(A, \Omega^1) = \mathfrak{a}^*$. Here $\mathfrak{a}$ is the Lie algebra of A. Let $\mathfrak{L}$ be the Lie algebra of the group L^0, so that we have the exact sequence

$$0 \to \mathfrak{a}^* \to \mathfrak{L}^* \xrightarrow{\pi} \mathbb{C} \to 0. \tag{5.2}$$

Denote by $\widetilde{L} = p^*L$ the pullback of L to L^0. For any $\nabla \in \mathrm{Conn}(L)$ let $\widetilde{\nabla}$ be its pullback to a connection in $\widetilde{L}$. Denote also by f the tautological section of $\widetilde{L}$ over L^0 (given by the identity map).

(5.3) Proposition. *The 1-forms $\widetilde{\nabla} \log f$, $\nabla \in \mathrm{Conn}(L)$, are invariant (with respect to the group structure on L^0). Their images in $\mathfrak{L}^*$ under the evaluation at 0 form the subspace $\pi^{-1}(1)$, where π is as in (5.2).*

Proof: Let us first show the invariance. Note that $\widetilde{\nabla} \log f$ is just the Lie algebra-valued 1-form on the total space of the principal $\mathbb{C}^*$-bundle L^0, describing the connection ∇ in L^0 in the standard approach of differential geometry. So its invariance follows from the fact that the line bundle (L, ∇) (or, what is equivalent, the $\mathbb{C}^*$-bundle (L^0, ∇)) satisfies the theorem on the square as a bundle with connection. More precisely, if $m, q_1, q_2 : A \times A \to A$ are the group structure and the two projections, then the isomorphism

$$\mu : q_1^* L \otimes q_2^* L \to m^* L$$

given by the theorem on the square is an isomorphism of bundles with connection. In particular, the induced isomorphism $\mu_a : L_a \otimes L \to l_a^* L$ is an isomorphism of bundles with connection on A. But the translation $l_{(a,\lambda)}$, $\lambda \in L_a \setminus \{0\}$ on L^0 is just given by $\mu_a(\lambda \otimes -)$. This shows the invariance. As for the second assertion of Proposition 5.3, it is again obvious from the interpretation of $\widetilde{\lambda} \log f$ as the Lie algebra-valued 1-form describing the connection and the identification $L_0^0 = \mathbb{C}^*$.

(5.4) Corollary. *The 1-forms $s\widetilde{\nabla} \log f$, $s \in \mathbb{C}^*$, $\nabla \in \mathrm{Conn}(L)$, form a nonempty Zariski open set in the space of all invariant 1-forms on L^0.*

Consider now several line bundles of degree 0, say $L_1, \dots, L_n$, on A, and let

$$0 \to (\mathbb{C}^*)^n \to G = L_1^0 \times_A \dots \times_A L_n^0 \xrightarrow{p} A \to 0 \tag{5.5}$$

be the associated semiabelian variety.

We denote $\widetilde{L}_i = p^* L_i$ and let $f_i \in H^0(G, \widetilde{L}_i)$ be the tautological section. As before, for $\nabla_i \in \mathrm{Conn}(L_i)$ we denote by $\widetilde{\nabla}_i$ its pullback to $\widetilde{L}_i$. Corollary 5.4 implies easily:

(5.6) Proposition. *Suppose $n \geq 0$. Then, the 1-forms $\sum_{i=1}^n s_i \widetilde{\nabla}_i \log f_i$ for $s_1, \dots, s_n \in \mathbb{C}^*$, $\nabla_i \in \mathrm{Conn}(L_i)$, form a Zariski open dense set in the space of all invariant 1-forms on G.*

We now turn to the proof of Theorem 1.3. Our approach is similar to [18], §2. First of all, it is known that any semiabelian variety has the form (5.5) which we assume. Next, we assume $n > 0$ since for $n = 0$ the group $G = A$ is compact and the theorem follows from (1.1) and from Proposition 2.2. We set $L_0 = \mathcal{O}_A$ and compactify G by embedding it into the relative projectivization

$$j : G \hookrightarrow \mathbb{P} := \mathbb{P}(L_0 \oplus ... \oplus L_n) \xrightarrow{\rho} A.$$

We can think of $\mathbb{P}$ as having homogeneous coordinates $(t_0 : ...; t_n)$ with t_i being not a function any more but rather a section of $\rho^* L_i$. Note that $\rho^* L_i$ has a flat connection induced from ∇_i, whose restriction to G is $\widetilde{\nabla}_i$. The variety $\mathbb{P}$ is the union of the relative affine charts $A_\nu = \bigoplus_{i \neq \nu} L_i$. Inside A_0, the complement of G is given by the condition that one of the tautological sections (still denoted by f_i) of the pullback of L_i, vanishes.

Take generic $\nabla_i \in \mathrm{Conn}(L_i)$ so that for a Zariski open, dense set of $s \in (\mathbb{C}^*)^n$ the 1-form $\omega_s = \sum s_i \widetilde{\nabla}_i \log f_i$ satisfies Proposition 2.2. Then, we mimic the arguments of Section 4 but in the relative situation, using Theorem 3.5. We find that

$$CC(Rj_* \mathfrak{F}) = \lim_{\substack{s \to 0 \\ s \in E}} \left(CC(\mathfrak{F}) + \sum_i s_i \widetilde{\nabla}_i \log f_i \right),$$

the limit being taken in $T^* \mathbb{P}$. After this, the proof is identical to the argument at the end of Section 4, and the theorem is proven.

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